

A Generalization of Fueter's Theorem in Dunkl-Clifford Analysis

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Abstract

In this paper we first offer an alternative approach to extend the original Fueter's Theorem in Dunkl-Clifford analysis to a version of the higher order case. Then this result is used to prove a generalized version of Fueter's Theorem with an extra homogeneous Dunkl-monogenic polynomial $P_n(x_0, \underline{x})$ in \mathbb{R}_1^d instead of the classical factor $P_n(\underline{x})$ in \mathbb{R}^d .

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1 Introduction

To explain the idea we start from the very basic facts. Let f be a holomorphic function in an open set of the upper half complex plane and write

$$f(z) = u(s, t) + \mathfrak{I}v(s, t), \quad z = s + \mathfrak{I}t,$$

then the Fueter's Theorem in Dunkl-Clifford analysis([7]) asserts that in the corresponding region there holds

$$D\Delta^{\gamma_\kappa + \frac{d-1}{2}} \left(u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|) \right) = 0$$

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whenever $\gamma_\kappa + (d-1)/2$ is any positive integer. Here $\underline{x} \in \mathbb{R}^d$, $D = T_{x_0} + \underline{D}$, $\underline{D} = \sum_{i=1}^d \mathbf{e}_i T_{x_i}$ and $\Delta = \sum_{i=0}^d T_{x_i}^2$, where T_{x_i} ($i = 0, 1, \dots, d$) are Dunkl operators corresponding to a finite reflection group W which leaves x_0 -axis invariant.

Furthermore, the authors in [7] generalized the above result as follows:

If, under the same assumptions of the above theorem, $P_n(\underline{x})$ is a homogeneous Dunkl-monogenic function of degree n with respect to the Dunkl-Dirac operator \underline{D} , i.e. $\underline{D}P_n(\underline{x}) = 0$, then

$$D\Delta^{\gamma_\kappa+n+\frac{d-1}{2}} \left(\left(u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|) \right) P_n(\underline{x}) \right) = 0$$

whenever $\gamma_\kappa + n + (d-1)/2$ is a positive integer.

The goal of this paper is to prove that the above result is still valid if we replace $P_n(\underline{x})$ by a homogeneous monogenic polynomial $P_n(x_0, \underline{x})$ of degree n in \mathbb{R}_1^d , which is a counterpart of generalized Fueter's Theorem([10]) in classical Clifford analysis. Motivated by [10], we will first prove an extension of Fueter's Theorem in Dunkl-Clifford analysis which uses complex-valued functions satisfying the following equation

$$\partial_{\bar{z}} \Delta_z^m f(z) = 0, \quad m \in \mathbb{N}_0, \quad (1)$$

as initial function instead of the usual holomorphic functions, where $z = t + \beta s$, $\partial_{\bar{z}}$ and Δ_z denote, respectively, the classical Cauchy-Riemann operator

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_t + \beta \partial_s)$$

and Laplace operator in two dimensions

$$\Delta_z = \partial_t^2 + \partial_s^2.$$

The current paper is organized as follows. Section 2 contains some basic facts about Dunkl-Clifford analysis and a characterization of Dunkl-Dirac operator in spherical coordinates. A higher order version of original Fueter's Theorem in Dunkl case is considered in section 3. In the last section we prove a version of generalized Fueter's Theorem with an extra Dunkl-monogenic factor $P_n(x_0, \underline{x})$.

2 Preliminaries and Dunkl-Clifford Analysis

Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be an orthonormal basis of \mathbb{R}^d satisfying the anti-commutation relationship $\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i = -2\delta_{ij}$, where δ_{ij} is the Kronecker symbol. We define the universal real-valued Clifford algebra $\mathbb{R}_{0,d}([2],[4])$ as the 2^d -dimensional associative algebra with basis given by $\mathbf{e}_0 = 1$ and $\mathbf{e}_A = \mathbf{e}_{l_1} \cdots \mathbf{e}_{l_n}$, where $A = \{l_1, \dots, l_n\} \subset \{1, \dots, d\}$, for $1 \leq l_1 < \dots < l_n \leq d$. Hence, each element $x \in \mathbb{R}_{0,d}$ will be represented by $x = \sum_A x_A \mathbf{e}_A$, $x_A \in \mathbb{R}$.

In what follows, $sc[x] = x_0$ will denote the scalar part of $x \in \mathbb{R}_{0,d}$, while an element $x = (x_0, x_1, \dots, x_d)$ of \mathbb{R}_1^d will be identified with $x = x_0 + \underline{x}$, $\underline{x} = \sum_{i=1}^d x_i \mathbf{e}_i$. Also, we need the anti-involution $\bar{\cdot}$ defined by $\bar{\mathbf{e}}_i = -\mathbf{e}_i$, and $\overline{\mathbf{e}_i \mathbf{e}_j} = \bar{\mathbf{e}}_j \bar{\mathbf{e}}_i$. An important property of algebra $\mathbb{R}_{0,d}$ is that each non-zero vector x in \mathbb{R}^d (or in \mathbb{R}_1^d) has a multiplicative inverse given by $\frac{\bar{x}}{|x|^2}$. A $\mathbb{R}_{0,d}$ -valued function f over $\Omega \subset \mathbb{R}_1^d$ has a representation $f = \sum_A \mathbf{e}_A f_A$ with component $f_A : \Omega \rightarrow \mathbb{R}$.

The reflection $\sigma_\alpha x$ of a given vector $x \in \mathbb{R}_1^d$ on the hyperplane H_α orthogonal to $\alpha \neq 0$ is given by

$$\sigma_\alpha x := x - 2 \frac{\langle \alpha, x \rangle}{|\alpha|^2} \alpha.$$

A finite set $R \subset \mathbb{R}_1^d \setminus \{0\}$ is called a root system if $R \cap \mathbb{R}_1^d \cdot \alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R the reflections σ_α , $\alpha \in R$, generate a finite group $W \subset O(d)$, called the finite reflection group (or Coxeter group) associated with R . All reflections in W correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d \setminus \bigcup_{\alpha \in R} H_\alpha$, we fix the positive subsystem $R_+ = \{\alpha \in R \mid \langle \alpha, \beta \rangle > 0\}$, i.e. for each $\alpha \in R$ either $\alpha \in R_+$ or $-\alpha \in R_+$. Sometimes we will only consider reflections which only act in \mathbb{R}^d . In this case we denote α or β by $\underline{\alpha}$ or $\underline{\beta}$. A function $\kappa : R \rightarrow \mathbb{R}^+$ on a root system R is called a multiplicity function if it is invariant under the action of the associated reflection group W . For abbreviation, we introduce the index $\gamma_\kappa = \sum_{\alpha \in R_+} \kappa(\alpha)$ and the Dunkl-dimension $\mu = 2\gamma_\kappa + d$.

For each fixed positive subsystem R_+ and multiplicity function κ we have, as invariant operators, the differential-difference operators (also called Dunkl operators) ([5],[6]):

$$T_{x_i} f(x) = \frac{\partial}{\partial x_i} f(x) + \sum_{\alpha \in R_+} \kappa(\alpha) \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle} \alpha_i, \quad i = 0, 1, \dots, d, \quad (2)$$

for $f \in C^1(\mathbb{R}_1^d)$. In the case $\kappa = 0$, the T_{x_i} , $i = 0, 1, \dots, d$, reduce to the corresponding partial derivatives. This also gives us the justification to think of these differential-difference operators as the equivalent of partial derivatives in the case of finite reflection groups. In this paper, we will assume throughout that $\kappa \geq 0$ and $\gamma_\kappa > 0$. More importantly, these operators mutually commute; that is, $T_{x_i}T_{x_j} = T_{x_j}T_{x_i}$. This property allows us to define a Dunkl-Dirac operator in \mathbb{R}^d for the corresponding reflection group W given by([3])

$$\underline{D}f = \sum_{i=1}^d \mathbf{e}_i T_{x_i} f. \quad (3)$$

So, the Dunkl Laplacian $\underline{\Delta}$ in \mathbb{R}^d is defined by $\underline{\Delta} = -\underline{D}^2 = \sum_{i=1}^d T_{x_i}^2$. We now introduce the Dunkl-Cauchy-Riemann operator in \mathbb{R}_1^d

$$D = T_{x_0} + \underline{D},$$

and Dunkl Laplacian in \mathbb{R}_1^d

$$\Delta = T_{x_0}^2 + \underline{\Delta}.$$

In this paper we will assume that our group W will leave the x_0 -axis invariant. Since in this case we have $T_{x_0} = \partial_{x_0}$ the Dunkl-Cauchy-Riemann operator and Dunkl-Laplacian in \mathbb{R}_1^d can also be written by

$$D = \partial_{x_0} + \underline{D}, \quad (4)$$

and

$$\Delta = \partial_{x_0}^2 + \underline{\Delta}. \quad (5)$$

Functions belonging to the kernel of Dunkl-Dirac operator \underline{D} or Dunkl-Cauchy-Riemann operator D will be called Dunkl-monogenic functions. As usual, functions belonging to be the kernel of Dunkl Laplacian will be called Dunkl-harmonic functions.

From [7] we have the following representation of the Dunkl-Dirac operator in spherical coordinates.

Proposition 2.1 *In spherical coordinates, i.e. $r = |\underline{x}|$ and $\underline{\omega} = \frac{\underline{x}}{|\underline{x}|}$ for $\underline{x} \in \mathbb{R}^d$, the Dunkl-Dirac operator has the form:*

$$\underline{D} = \underline{\omega}(\partial_r + \frac{1}{r}\Gamma_{\underline{\omega}}), \quad (6)$$

with

$$\Gamma_{\underline{\omega}} = \gamma_{\kappa} + \Phi_{\underline{\omega}} + \Psi,$$

where

$$\Phi_{\underline{\omega}} = - \sum_{i < j} \mathbf{e}_i \mathbf{e}_j (x_i \partial_{x_j} - x_j \partial_{x_i}),$$

and

$$\Psi f(\underline{x}) = - \sum_{i < j} \mathbf{e}_i \mathbf{e}_j \sum_{\underline{\alpha} \in R^+} \kappa(\underline{\alpha}) \frac{f(\underline{x}) - f(\sigma_{\underline{\alpha}} \underline{x})}{\langle \underline{\alpha}, \underline{x} \rangle} (x_i \alpha_j - x_j \alpha_i) - \sum_{\underline{\alpha} \in R^+} \kappa(\underline{\alpha}) f(\sigma_{\underline{\alpha}} \underline{x}),$$

for any $f \in C^1(\mathbb{R}^d)$.

Remark 2.1 The operator $\Phi_{\underline{\omega}}$ in above proposition corresponds to the classical spherical vector derivatives (the classic Gamma operator) while the additional operator Ψ and constant γ_{κ} derive from the difference part of Dunkl operators.

Furthermore, $\Gamma_{\underline{\omega}}$ is a first differential-difference operator which satisfies the following properties([7]):

Proposition 2.2 Assume that $f(x) = f(|\underline{x}|)$ is a radial function and $P_n(\underline{x})$ is a homogeneous Dunkl-monogenic function of degree n , $n \in \mathbb{R}$, then

- (i) $\Gamma_{\underline{\omega}} f(r) = 0$,
- (ii) $\Gamma_{\underline{\omega}} P_n(\underline{\omega}) = -n P_n(\underline{\omega})$,
- (iii) $\Gamma_{\underline{\omega}}(\underline{\omega} P_n(\underline{\omega})) = (\mu + n - 1) \underline{\omega} P_n(\underline{\omega})$.

Henceforward, we denote by $\mathcal{M}(n)$ the space of all homogeneous Dunkl-monogenic polynomials of degree $n \in \mathbb{N}$. We then immediately have

Lemma 2.1 Let $k \in \mathbb{N}$ and $P_n \in \mathcal{M}(n)$. Then there has

$$D(\underline{x}^k P_n(\underline{x})) = \begin{cases} -k x^{k-1} P_n(x), & k \text{ even}, \\ -(k + \mu + 2n - 1) x^{k-1} P_n(x), & k \text{ odd}. \end{cases}$$

3 A higher order version of the original Fueter's Theorem

In this section we prove an extension of the original Fueter's Theorem in Dunkl case starting from a complex-valued function satisfying equation (1) instead of the usual holomorphic function. To this end, we start with the following lemma from [9], [8] or [10] which we only state the special case that we will use in this paper.

Lemma 3.1 *Suppose that $f(x_0, r)$ is a scalar-valued infinitely differentiable functions in \mathbb{R}^2 and that D_r and D^r are differential operators defined by $D_r(0)\{f\} = D^r(0)\{f\} = f$ and*

$$D_r(m)\{f\} = \left(\frac{1}{r}\partial_r\right)^m \{f\},$$

$$D^r(m)\{f\} = \partial_r \left(\frac{D^r(m-1)\{f\}}{r} \right)$$

for integer $m \geq 1$. Then one has

- (i) $\partial_r^2 D_r(m)\{f\} = D_r(m)\{\partial_r^2 f\} - 2m D_r(m+1)\{f\},$
- (ii) $\partial_r^2 D^r(m)\{f\} = D^r(m)\{\partial_r^2 f\} - 2m D^r(m+1)\{f\},$
- (iii) $D^r(m)\{\partial_r f\} = \partial_r D_r(m)\{f\},$
- (iv) $D_r(m)\{\partial_r f\} - \partial_r D^r(m)\{f\} = 2m/r D^r(m)\{f\}.$

Furthermore, we need the following lemma which shows that the iterated Dunkl-Laplacian Δ^m , for any positive integer m , keeps functions of the form $(f(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} g(x_0, |\underline{x}|)) P_n(\underline{x})$ invariant whenever f and g are scalar-valued functions in \mathbb{R}^2 and $P_n(\underline{x})$ be a homogeneous Dunkl-monogenic function of degree n in \mathbb{R}^d .

Lemma 3.2 *Let $f(x_0, r)$ be a scalar-valued infinitely differentiable function in an open set of $\mathbb{R}_+^2 = \{(t, s) \in \mathbb{R}^2 : s > 0\}$. Then for $m \in \mathbb{N}_0$ we have*

$$\Delta^m(f(x_0, |\underline{x}|) P_n(\underline{x})) = \left(\sum_{j=0}^m d_{n,\mu}(j) \binom{m}{j} D_r(j) \{ \Delta_z^{m-j} f(x_0, r) \} \right) P_n(\underline{x})$$

and

$$\Delta^m(f(x_0, |\underline{x}|) \frac{\underline{x}}{|\underline{x}|} P_n(\underline{x})) = \left(\sum_{j=0}^m d_{n,\mu}(j) \binom{m}{j} D^r(j) \{ \Delta_z^{m-j} f(x_0, r) \} \right) \frac{\underline{x}}{|\underline{x}|} P_n(\underline{x}),$$

where

$$\begin{aligned} d_{n,\mu}(0) &= 1, \\ d_{n,\mu}(j) &= (2n + \mu - 1)(2n + \mu - 3) \cdots (2n + \mu - (2j - 1)). \end{aligned}$$

Proof: We will prove this lemma by induction. Let $\underline{\omega} = \frac{\underline{x}}{|\underline{x}|}$. When $m = 1$, we need to show that the following identities hold

$$\Delta(f P_n) = (\Delta_z f + (2n + \mu - 1) D_r(1) \{f\}) P_n \quad (7)$$

and

$$\Delta(f \underline{\omega} P_n) = (\Delta_z f + (2n + \mu - 1) D^r(1) \{f\}) \underline{\omega} P_n. \quad (8)$$

To prove (7), we start from

$$\Delta_h = \partial_{x_0}^2 - \underline{D}_h \underline{D}_h, \quad \underline{D}_h = \underline{\omega} (\partial_r + \frac{1}{r} \Gamma_{\underline{\omega}}).$$

Then using Proposition 2.2 we get

$$\begin{aligned} \underline{D}(f P_n) &= \underline{\omega} (\partial_r + \frac{1}{r} \Gamma_{\underline{\omega}}) (f r^n P_n(\underline{\omega})) \\ &= (\partial_r f) r^n \underline{\omega} P_n(\underline{\omega}) \end{aligned}$$

and

$$\begin{aligned} \underline{D} \underline{D}(f P_n) &= \underline{\omega} (\partial_r + \frac{1}{r} \Gamma_{\underline{\omega}}) ((\partial_r f) r^n \underline{\omega} P_n(\underline{\omega})) \\ &= -(\partial_r^2 f) r^n P_n(\underline{\omega}) - n(\partial_r f) r^{n-1} P_n(\underline{\omega}) \\ &\quad - (\mu + n - 1) (\partial_r f) r^{n-1} P_n(\underline{\omega}) \\ &= - \left(\partial_r^2 f + \frac{2n + \mu - 1}{r} (\partial_r f) \right) r^n P_n(\underline{\omega}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \Delta(f P_n) &= \left(\partial_{x_0}^2 f + \partial_r^2 f + \frac{2n + \mu - 1}{r} (\partial_r f) \right) r^n P_n(\underline{\omega}) \\ &= (\Delta_z f + (2n + \mu - 1) D_r(1) \{f\}) P_n. \end{aligned}$$

To prove (8), again applying Proposition 2.2 we obtain

$$\begin{aligned}
\underline{D}(f\underline{\omega}P_n) &= \underline{\omega}(\partial_r + \frac{1}{r}\Gamma_{\underline{\omega}})(fr^n\underline{\omega}P_n(\underline{\omega})) \\
&= \underline{\omega}((\partial_r f)r^n\underline{\omega}P_n(\underline{\omega}) + nfr^{n-1}\underline{\omega}P_n(\underline{\omega}) + fr^{n-1}\Gamma_{\underline{\omega}}(\underline{\omega}P_n(\underline{\omega}))) \\
&= -(\partial_r f)r^nP_n(\underline{\omega}) - (2n + \mu - 1)fr^{n-1}P_n(\underline{\omega})
\end{aligned}$$

and

$$\begin{aligned}
&\underline{DD}(f\underline{\omega}P_n) \\
&= -\underline{\omega}(\partial_r + \frac{1}{r}\Gamma_{\underline{\omega}})((\partial_r f)r^nP_n(\underline{\omega}) + (2\gamma_\kappa + 2n + d - 1)fr^{n-1}P_n(\underline{\omega})) \\
&= -\underline{\omega}((\partial_r^2 f)r^n + n(\partial_r f)r^{n-1})P_n(\underline{\omega}) + (\partial_r f)r^{n-1}\Gamma_{\underline{\omega}}(P_n(\underline{\omega})) \\
&\quad + (2n + \mu - 1)((\partial_r f)r^{n-1} + (n - 1)fr^{n-2})P_n(\underline{\omega}) \\
&\quad + (2n + \mu - 1)fr^{n-2}\Gamma_{\underline{\omega}}(P_n(\underline{\omega})) \\
&= -\left(\partial_r^2 f + \frac{n}{r}(\partial_r f) - \frac{n}{r}(\partial_r f) + \frac{2n + \mu - 1}{r}(\partial_r f) \right. \\
&\quad \left. + \frac{(2n + \mu - 1)(n - 1)}{r^2}f - \frac{(2n + \mu - 1)n}{r^2}f\right)\underline{\omega}r^nP_n(\underline{\omega}) \\
&= -\left(\partial_r^2 f + (2n + \mu - 1)\left(\frac{\partial_r f}{r} - \frac{f}{r^2}\right)\right)\underline{\omega}P_n.
\end{aligned}$$

This leads to

$$\begin{aligned}
\Delta(f\underline{\omega}P_n) &= \left(\partial_{x_0}^2 f + \partial_r^2 f + (2n + \mu - 1)\left(\frac{\partial_r f}{r} - \frac{f}{r^2}\right)\right)\underline{\omega}P_n \\
&= ((\Delta_z f + (2n + \mu - 1)D^r(1)\{f\})\underline{\omega})P_n.
\end{aligned}$$

Summarizing we have that the lemma is true in the case $m = 1$. Assume that our formulae hold for a positive integer m , we have to show them for $m + 1$.

We thus get

$$\begin{aligned}
\Delta^{m+1}(fP_n) &= \sum_{j=0}^m d_{n,\mu}(j) \binom{m}{j} \Delta(D_r(j)\{\Delta_z^{m-j}f\}P_n) \\
&= \sum_{j=0}^m d_{n,\mu}(j) \binom{m}{j} (\partial_{x_0}^2 D_r(j)\{\Delta_z^{m-j}f\} + \partial_r^2 D_r(j)\{\Delta_z^{m-j}f\})
\end{aligned}$$

$$\begin{aligned}
& + \frac{2n + \mu - 1}{r} \partial_r D_r(j) \{ \Delta_z^{m-j} f \} P_n \\
& = \sum_{j=0}^m d_{n,\mu}(j) \binom{m}{j} (D_r(j) \{ \partial_{x_0}^2 \Delta_z^{m-j} f + \partial_r^2 \Delta_z^{m-j} f \} \\
& \quad - 2j D_r(j+1) \{ \Delta_z^{m-j} f \} + (2n + \mu - 1) D_r(j+1) \Delta_z^{m-j} f) P_n \\
& = \left(\sum_{j=0}^m d_{n,\mu}(j) \binom{m}{j} D_r(j) \{ \Delta_z^{m+1-j} f \} \right) P_n \\
& \quad + \left(\sum_{j=0}^m d_{n,\mu}(j) \binom{m}{j} (2n + \mu - (2j + 1)) D_r(j+1) \{ \Delta_z^{m-j} f \} \right) P_n \\
& = \left(\sum_{j=0}^{m+1} d_{n,\mu}(j) \binom{m+1}{j} D_r(j) \{ \Delta_z^{m+1-j} f \} \right) P_n.
\end{aligned}$$

In the last step we used the fact that $\binom{m}{j} + \binom{m}{j-1} = \binom{m+1}{j}$. This establishes the first formula. The other one may be proved in a similar way. ■

Now we arrive at the following higher order version of Fueter's Theorem in Dunkl case.

Theorem 3.1 *Let $f(z) = u(t, s) + \mathfrak{B}v(t, s)$, $z = t + \mathfrak{B}s$, be a complex-valued function satisfying equation (1) in some open set $\Omega \subset \mathbb{C}^+ = \{z \in \mathbb{C} : s > 0\}$. If μ is odd, then the following function*

$$\Delta^{m+n+\frac{\mu-1}{2}} \left(\left(u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|) \right) P_n(\underline{x}) \right)$$

is Dunkl-monogenic in $\vec{\Omega} = \{x \in \mathbb{R}_1^d : (x_0, |\underline{x}|) \in \Omega\}$.

Proof: From Lemma 3.2, we get that

$$\begin{aligned}
& \Delta^{m+n+\frac{\mu-1}{2}} (u(x_0, |\underline{x}|) P_n(\underline{x})) \\
& = \left(\sum_{j=0}^{m+n+\frac{\mu-1}{2}} d_{n,\mu}(j) \binom{m+n+\frac{\mu-1}{2}}{j} D_r(j) \{ \Delta_z^{m+n+\frac{\mu-1}{2}-j} u(x_0, r) \} \right) P_n(\underline{x})
\end{aligned}$$

and

$$\begin{aligned} & \Delta^{m+n+\frac{\mu-1}{2}}(v(x_0, |\underline{x}|) \frac{\underline{x}}{|\underline{x}|} P_n(\underline{x})) \\ &= \left(\sum_{j=0}^{m+n+\frac{\mu-1}{2}} d_{n,\mu}(j) \binom{m+n+\frac{\mu-1}{2}}{j} D^r(j) \{ \Delta_z^{m+n+\frac{\mu-1}{2}-j} v(x_0, r) \} \right) \frac{\underline{x}}{|\underline{x}|} P_n(\underline{x}). \end{aligned}$$

Obviously, the first $n + (\mu - 1)/2$ terms in the above equalities vanish since by hypothesis f satisfies equation (1). Furthermore, note that $2n + \mu - (2j - 1) < 0$ for $j \geq n + (\mu + 1)/2$ and therefore $d_{n,\mu}(j) = 0$ for $j \geq n + (\mu + 1)/2$.

Summarizing we obtain that

$$\begin{aligned} & \Delta^{m+n+\frac{\mu-1}{2}} \left(\left(u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|) \right) P_n(\underline{x}) \right) \\ &= (2n + \mu - 1)!! \binom{m+n+\frac{\mu-1}{2}}{n+\frac{\mu-1}{2}} (A(x_0, r) + \frac{\underline{x}}{|\underline{x}|} B(x_0, r)) P_n(\underline{x}), \end{aligned}$$

where

$$\begin{aligned} A &= D_r \left(n + \frac{\mu - 1}{2} \right) \{ \Delta_z^m u \}, \\ B &= D^r \left(n + \frac{\mu - 1}{2} \right) \{ \Delta_z^m v \}. \end{aligned}$$

The task is now to show that A and B satisfy the following Vekua-type system in Dunkl case([7])

$$\begin{cases} \partial_{x_0} A - \partial_r B = \frac{2n+\mu-1}{r} B, \\ \partial_{x_0} B + \partial_r A = 0. \end{cases}$$

To this end it will be necessary to use the assumptions on u and v

$$\begin{cases} \partial_t \Delta_z^m u - \partial_s \Delta_z^m v = 0, \\ \partial_t \Delta_z^m v + \partial_s \Delta_z^m u = 0, \end{cases}$$

and statements (iii) and (iv) of Lemma 3.1.

Indeed, we have

$$\begin{aligned}
\partial_{x_0} A - \partial_r B &= D_r \left(n + \frac{\mu-1}{2} \right) \{ \partial_{x_0} \Delta_z^m u \} - \partial_r D^r \left(n + \frac{\mu-1}{2} \right) \{ \Delta_z^m v \} \\
&= D_r \left(n + \frac{\mu-1}{2} \right) \{ \partial_r \Delta_z^m v \} - \partial_r D^r \left(n + \frac{\mu-1}{2} \right) \{ \Delta_z^m v \} \\
&= \frac{2n + \mu - 1}{r} D^r \left(n + \frac{\mu-1}{2} \right) \{ \Delta_z^m v \} \\
&= \frac{2n + \mu - 1}{r} B
\end{aligned}$$

and

$$\begin{aligned}
\partial_{x_0} B + \partial_r A &= D^r \left(n + \frac{\mu-1}{2} \right) \{ \partial_{x_0} \Delta_z^m v \} + \partial_r D_r \left(n + \frac{\mu-1}{2} \right) \{ \Delta_z^m u \} \\
&= D^r \left(n + \frac{\mu-1}{2} \right) \{ \partial_{x_0} \Delta_z^m v \} + D^r \left(n + \frac{\mu-1}{2} \right) \{ \partial_r \Delta_z^m u \} \\
&= D^r \left(n + \frac{\mu-1}{2} \right) \{ \partial_{x_0} \Delta_z^m v + \partial_r \Delta_z^m u \} \\
&= 0,
\end{aligned}$$

which completes the proof. \blacksquare

4 Fueter's Theorem with an extra Dunkl-monogenic factor $P_n(x_0, \underline{x})$

We begin this section with two basic theorems of Dunkl-Clifford analysis. The proof of Theorem 4.1 is straightforward from the one of CK-Extension Theorem in classical Clifford analysis. Theorem 4.2 is from [11] or [1].

Theorem 4.1 *Any analytic function $f(\underline{x})$ in \mathbb{R}^d has a unique Dunkl-monogenic extension $CK[g]$ to \mathbb{R}_1^d , which is given by*

$$CK[g(\underline{x})](x) = \sum_{j=0}^{\infty} \frac{(-x_0)^j}{j!} \underline{D}^j g(\underline{x}).$$

Let $\mathcal{P}(n)$, $n \in \mathbb{N}$, denote the set of all $\mathbb{R}_{0,d}$ -valued homogeneous polynomials of degree n in \mathbb{R}^d , which contain the important subspace $\mathcal{M}(n)$ introduced in Section 2. Then there holds the following Almasi-Fischer Decomposition Theorem.

Theorem 4.2 *Let $n \in \mathbb{N}$. Then*

$$\mathcal{P}(n) = \bigoplus_{k=0}^n \underline{x}^k \mathcal{M}(n-k).$$

Now we are ready to prove the following Fueter's Theorem in Dunkl-Clifford analysis with an extra Dunkl-monogenic factor $P_n(x_0, \underline{x})$.

Theorem 4.3 *Let $f(z) = u(t, s) + \mathfrak{B}v(t, s)$ be a complex-valued holomorphic function in some open set $\Omega \subset \mathbb{C}^+$ and assume that $P_n(x_0, \underline{x})$ is a homogeneous Dunkl-monogenic polynomial of degree n in \mathbb{R}_1^d . If μ is odd, then the following function*

$$\Delta^{n+\frac{\mu-1}{2}} \left(\left(u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|) \right) P_n(x_0, \underline{x}) \right)$$

is Dunkl-monogenic in $\vec{\Omega} = \{x \in \mathbb{R}_1^d : (x_0, |\underline{x}|) \in \Omega\}$.

Proof: It is obvious from Theorem 4.1 that $P_n(x_0, \underline{x}) = CK[P_n(0, \underline{x})](x)$. By Theorem 4.2 there exist unique $P_{n-k} \in \mathcal{M}(n-k)$ such that

$$P_n(x_0, \underline{x}) = \sum_{k=0}^n CK[\underline{x}^k P_{n-k}(\underline{x})](x).$$

Thus, it suffices to show that the Dunkl-monogenicity of function

$$\Delta^{n+\frac{\mu-1}{2}} \left(\left(u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|) \right) CK[\underline{x}^k P_{n-k}(\underline{x})](x) \right), \quad k = 0, \dots, n.$$

Since

$$CK[\underline{x}^k P_{n-k}(\underline{x})](x) = \sum_{j=0}^n \frac{(-x_0)^j}{j!} \underline{D}^j(\underline{x}^k P_{n-k}(\underline{x}))$$

and by means of Lemma 2.1, we can conclude that $CK[\underline{x}^k P_{n-k}(\underline{x})](x)$ is of the form

$$CK[\underline{x}^k P_{n-k}(\underline{x})](x) = \left(\sum_{j=0}^k c_j x_0^j \underline{x}^{k-j} \right) P_{n-j}(\underline{x}), \quad c_j \in \mathbb{R}.$$

Therefore,

$$CK[\underline{x}^k P_{n-k}(\underline{x})](x) = \left(U(x_0, r) + \frac{\underline{x}}{|\underline{x}|} V(x_0, r) \right) P_{n-k}(\underline{x}),$$

where U and V are real-valued homogeneous polynomials of degree k in the variables x_0 and r . So, its corresponding complex-valued function $g(z) = U(t, s) + \beta V(t, s)$ obviously satisfies

$$\partial_{\bar{z}}^{k+1} g(z) = 0, \quad z = t + \beta s \in \mathbb{C}.$$

Whence by the assumption of f ,

$$\partial_{\bar{z}}^{k+1} (f(z)g(z)) = 0, \quad z \in \Omega,$$

i.e. $f(z)g(z)$ is $(n+1)$ -holomorphic in Ω . It then follows that

$$\partial_{\bar{z}} \Delta_z^k (f(z)g(z)) = 0, \quad z \in \Omega.$$

The proof of Theorem 4.3 now follows by using Theorem 3.1. ■

Remark 4.1 *Comparing with the classical case(see [10]), the crucial part in our treatment of Theorem 4.3 is the replacement of the classical Euclidean dimension d by the Dunkl-dimension μ .*

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